# Year 13 Mathematics

# EAS 3.13

## Probability

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# Probability 3.13

This achievement standard involves applying probability concepts in solving problems.

| Achievement | Achievement with Merit | Achievement with Excellence |
|---|---|---|
| • Apply probability concepts in solving problems. | • Apply probability concepts, using relational thinking, in solving problems. | • Apply probability concepts, using extended abstract thinking, in solving problems. |

- This achievement standard is derived from Level 8 of The New Zealand Curriculum and is related to the achievement objective:
  - Investigate situations that involve elements of chance
    - calculating probabilities of independent, combined, and conditional events.
- Apply probability concepts in solving problems involves:
  - selecting and using methods
  - demonstrating knowledge of concepts and terms
  - communicating using appropriate representations.
- Relational thinking involves one or more of:
  - selecting and carrying out a logical sequence of steps
  - connecting different concepts or representations
  - demonstrating understanding of concepts

  and also relating findings to a context, or communicating thinking using appropriate statements.
- Extended abstract thinking involves one or more of:
  - devising a strategy to investigate or solve a problem
  - identifying relevant concepts in context
  - developing a chain of logical reasoning
  - making a generalisation

  and also where appropriate, using contextual knowledge to reflect on the answer.
- Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or statistical contexts.
- Methods include a selection from those related to:
  - true probability versus model estimates versus experimental estimates
  - randomness
  - independence
  - mutually exclusive events
  - conditional probabilities
  - probability distribution tables and graphs
  - two way tables
  - probability trees
  - Venn diagrams.

# Probability Concepts

## Language of Probability

We use the word probability to represent the chance of an event occurring, based on some evidence. If we know the probability of an event occurring we can use this probability to predict the probability of future events.

We begin by identifying some basic **probability** facts.

- all probabilities are assigned a numerical value between 0 and 1 inclusive.
- an impossible event has a probability of 0.
- an event which is a certainty has a probability of 1.
- we denote the probability of an event A occurring, by using the notation P(A).
- the list of possible outcomes is called the **sample space.**

To calculate the probability of an event occurring we begin by considering an 'experiment', namely tossing a die.

The possible outcomes (**sample space**) when tossing a die are {1, 2, 3, 4, 5, 6}, which we shall call E. The number of these equally likely outcomes in E is 6, written

$$n(E) = 6.$$

If we wish to calculate the probability of scoring a 3 or 4 when throwing a die (we refer to this event as event A) then A = {3, 4} and n(A) = 2.

$$\text{Therefore } P(A) = \frac{n(A)}{n(E)}$$

$$= \frac{2}{6}$$

Formally, if E is the set of all possible equally likely outcomes of an experiment, then the probability of an event A occurring is

$$P(A) = \frac{n(A)}{n(E)}$$

we often restate this as

$$P(A) = \frac{\text{Number of successful outcomes}}{\text{Total number of possible outcomes}}$$

where we have prior knowledge of possible outcomes and the physical situation allows us to deduce or infer the relevant probabilities.

e.g. P(drawing a heart from a pack of 52 cards) = $\frac{1}{4}$

Extra Notes

## True versus Experimental Probability cont...

For example, if we were trying to find the probability that two out of five students will get a ticket for not wearing a helmet, we are using a model that assumes that one result does not depend upon another (is independent).

Similarly once the police have issued one ticket the probability that they issue others in reality could change and not be constant.

If we based the result on an experiment in Wellington it may not be valid for other parts of New Zealand.

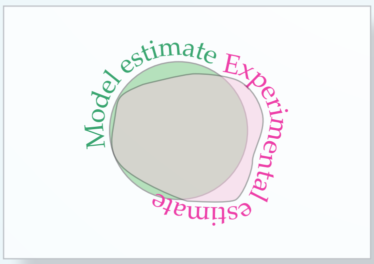
Another example of using an experimental probability to solve true problems is where we simulate the problem. If we run the simulation a large number of times we assume the relative frequency of each outcome will be the same as the true probability. Throwing a coin 10 000 times and finding we get 5065 heads may mean the probability of getting a head is 0.5065 (caused by uneveness of the metal on the sides of the coin) or more likely it could be because of chance and another experiment would give a result close to 0.5.

The differences between experimental probability and the true probability depends on the assumptions we have made and on chance as each time we run the simulation we will get slightly different results.

## True versus Theoretical Probability

Sometimes we calculate a probability by modelling the situation using theory. We look at all the possible outcomes and the likelihood of each and model this. It could be how often two out of three coins land heads up. We start with the assumption that the outcome (heads or tails) is equally likely and that one coin landing heads up does not affect the next coin. The first assumption may not be correct as the uneveness of the metal on the coin could affect the result or how the coin is tossed may also affect the result. Therefore our model may be close to but not a perfect representation of the true situation.

If we now look at the Venn diagram showing the true situation and the model probability you can see they could be different.




**In statistics many words have precise meanings so we need to learn their definition.**

| | |
|---|---|
| **True Probability** | **Usually never known. This is the actual probability of an event occurring.** |
| **Experimental Probability** | **Is an estimate of the probability based on a number of successful or true outcomes divided by the number of trials. This is also called relative frequency.** |
| **Model Probability** | **A model is based on an analysis of the situation looking at all possible outcomes and the likelihood of each outcome.** |

Extra Notes

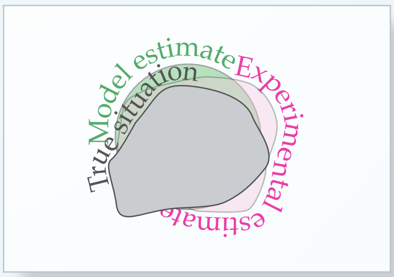
**To respond to questions about True vs Experimental vs Model Probability we need to state all assumptions.**

**Assumptions include:**

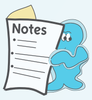
- **whether the true probability varies over time**
- **whether one result affects a subsequent result (independence)**
- **whether the experiment accurately represents the true situation or the model.**

**We also need to acknowledge that all experiments rely upon chance.**

If we were to now simulate this model to find the probability of two out of three coins landing heads up we may find the experimental probability is a little different from both the true situation and the model probability.



# Tree Diagrams



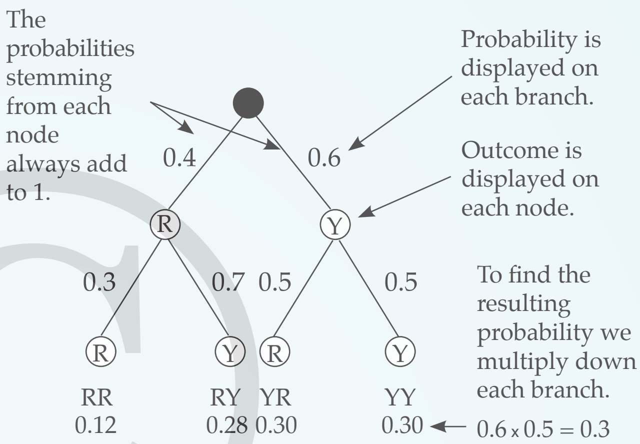

## Tree Diagrams

Some probability problems can be more easily solved by using **Tree Diagrams**. A tree diagram displays all the possible outcomes of a probability experiment. On each branch of the tree appropriate probabilities are assigned.

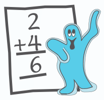
Tree diagrams can be drawn from left to right or top to bottom. In this text we have usually drawn probability trees from top to bottom.

The basic structure of a tree diagram is shown in the diagram on the right.

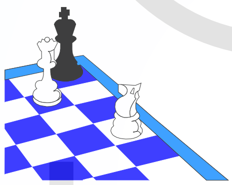
To find the final probabilities for an outcome we multiply together, from top to bottom, the probabilities on the branches for the required path. A good check is that the final probabilities must add to one.

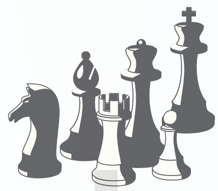



The probabilities of the final outcomes also add to 1, (0.12 + 0.28 + 0.30 + 0.30 = 1).




## Example



A man plays a two game chess tournament. For the first game his probabilities of winning, losing and drawing are 0.6, 0.3, 0.1.

- If he wins the first game his probabilities change to 0.7, 0.2, 0.1.
- If he loses they change to 0.5, 0.2, 0.3.
- If he draws they remain the same.

He scores one point for a win, half a point for a draw and zero points for a loss.

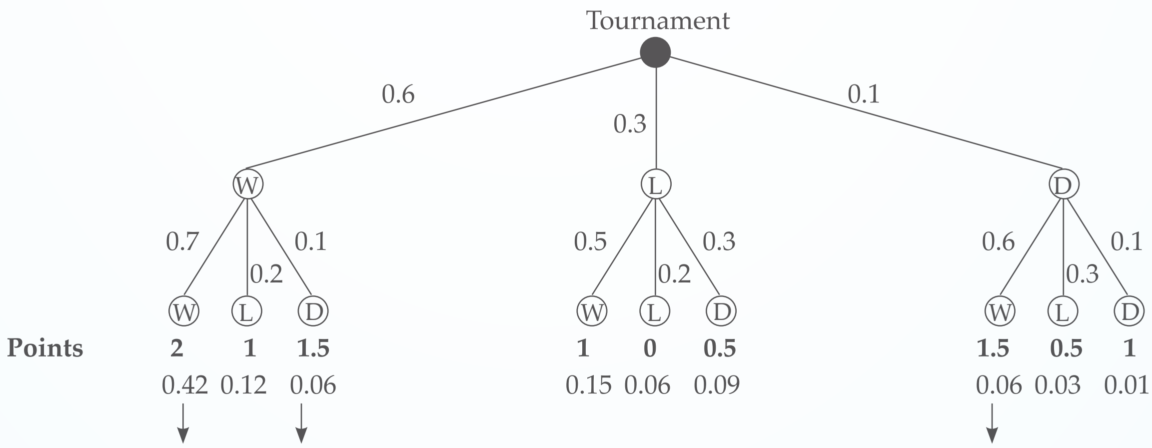
What is the probability of him scoring one and a half or more points?

We begin by drawing a tree diagram to represent the two game tournament and all the possible outcomes (see the diagram below).

Note: All the respective probabilities have been assigned to each branch and each group of win, lose and draw probabilities total to 1.

Underneath the tree diagram we have assigned the total number of points the man would have accumulated if he had obtained those results.

We then calculate the probabilities for the required route(s) and add them.




Prob. 0.42 + 0.06 + 0.06

P(scoring ≥ 1.5 points) = 0.42 + 0.06 + 0.06

= 0.54

**37.** From a pack of 52 playing cards, two are selected at random without replacement. Find the probability that

a) the second card is red, given that the first is black.

b) the second card is an ace, given that the first is not an ace.

**38.** On par-three golf holes, the probability that a PGA professional golfer hits his first shot on the green is 0.8.

The probability that a PGA professional golfer hits this first shot on the green and requires only one putt to put the ball in the hole is 0.2.

Given a PGA professional golfer hits his first shot on the green, find the probability that he only requires one putt to put the ball in the hole.

**39.** The table below gives the sex and qualifications of a group of students.

| | NCEA 1 | NCEA 2 | NCEA 3 | Total |
|---|---|---|---|---|
| Male | 83 | 72 | 52 | 207 |
| Female | 102 | 75 | 91 | 268 |
| Total | 185 | 147 | 143 | 475 |

If a student is selected at random find the probability

a) the student is a male given he has NCEA 2.

b) the student has NCEA 3 given they are male.

c) the student is female given that they have NCEA 1.

**40.** 48% of the students ride the bus to and from school. 30% of the students ride the bus and buy their lunch at school. Given a student rides the bus, find the probability that the student buys their lunch at school.

**41.** Events A, B and C satisfy the following conditions.

$P(A) = 0.6$, $P(B) = 0.8$, $P(B \mid A) = 0.45$ and $P(B \cap C) = 0.28$.

Calculate the following probabilities.

a) $P(A \cap B)$

b) $P(C \mid B)$

c) $P(A \mid B)$

**42.** A coin is flipped three times, what is the probability it comes up tails at least once given

a) all three flips produce the same result?

b) the third flip is heads?

**43.** A jar contains six red marbles and three white marbles. Two marbles are selected at random without replacement. Find the probability that the second marble selected is red given the first was red.

# Two Way Tables

## Two Way Tables

Often data is presented as a table where the results are split by another criteria such as the sex of the respondents.

| Survey of Year 10 | Male | Female |
|---|---|---|
| Hmwk. done | 45 | 52 |
| No homework | 12 | 11 |

If totals are not shown we should calculate the total number in each category.

| Survey of Year 10 | Male | Female | Totals |
|---|---|---|---|
| Hmwk. done | 45 | 52 | 97 |
| No homework | 12 | 11 | 23 |
| Totals | 57 | 63 | 120 |

In this example we can see that there are 57 male students and 63 female students, a total of 120 students. Further we can see that 97 students completed their homework and 23 did not.

We should be able to use a two way table to calculate all probabilities. The probability that a random student has done their homework is

$$\text{P(Hmwk.)} = \frac{97}{120}(0.8083)$$

The number in each cell represents the number who fulfilled criteria 1 AND criteria 2. There are 45 students who have done their homework AND are male. The probability that a random student is male and has done their homework is

$$\text{P(Hmwk. and male)} = \frac{45}{120}(0.3750)$$

A two way table makes conditional probability easy. If we wanted to find the probability that a random student completed their homework given they were male then we can see that 45 out of the 57 male students did their homework.

$$\text{P(Hmwk.} \mid \text{male)} = \frac{45}{57}(0.7895)$$

Similarly if we asked for the probability of a random student being female given they did their homework we would focus on the row in the table of students that completed their homework.

$$\text{P(Female} \mid \text{hmwk.)} = \frac{52}{97}(0.5361)$$

Extra Notes

**If the table is expressed in terms of probabilities, percentages or raw data check that the figures are consistent. Percentages should add to 100 and probabilities should add to 1. Similarly the total of all the cells should be the total amount of data. If the table is not consistent it could be that some responses were deliberately ignored (maybe they did not make sense) or there is an error in putting the table together. Blank or wrong answers etc. should be removed from a survey so tables are consistent.**

A two way table can have the original data or be expressed in terms of percentages or probabilities. The original table could have been presented as

| Survey of Year 10 | Male | Female |
|---|---|---|
| Hmwk. done | 37.5% | 43.3% |
| No homework | 10.0% | 9.2% |

We have lost the total figure but we can still calculate probabilities. The probability that a random student is female given they have done their homework would become

$$\text{P(Female} \mid \text{hmwk.)} = \frac{43.3}{80.8}(0.5359)$$

The slight difference with the earlier figure is caused by round off error.

Similarly the probability that a random female student has done their homework is

$$\text{P(Hmwk.} \mid \text{female)} = \frac{43.3}{52.5}(0.8248)$$

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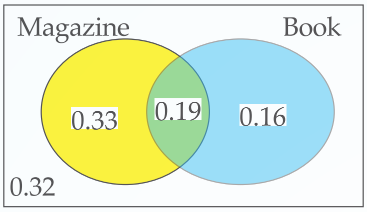
**48.** a)




b) P(book and mag) = 0.19

c) P(mag and not book) = 0.33

d) P(lotto, book and mag) = 0.1235

**49.** a) P(two successive tickets) = 0.0132

b) P(1 prize on either 2 tickets) = 0.2036

c) P(no prize on three tickets) = 0.6932

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**50.** a) P(at least one subject) = 0.902

b) P(only 1 of the 2 subjects) = 0.434

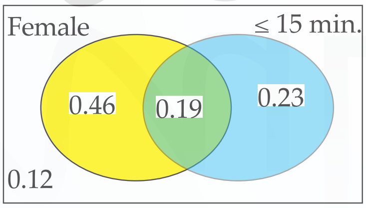
c) P(neither of the two subjects) = 0.098

**51.** a) P(away and win) = 0.13

b) P(lose a home game) = 0.18

c) P(away and lose) = 0.27

**52.** a)




b) P(male and ≤ 15 min) = 0.23

c) P(male and requires ≥ 15 min) = 0.12

d) P(female or requires ≤ 15 min) = 0.88

**53.** a) 120 students

b) P(statistics and English) = 0.075

c) P(all three subjects) = 0.0417

d) P(English but not biology) = 0.2417

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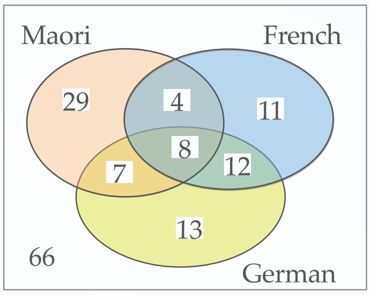
**54.** a)

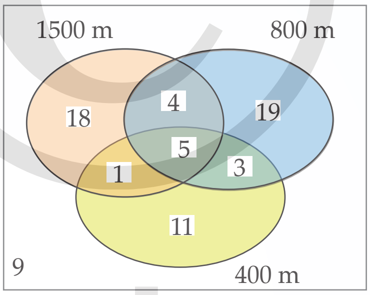



b) P(not study a language) = 0.44

c) P(at least two languages) = 0.2067

d) % study language, study French = 41.7%

**55.** a)




b) 9 athletes

c) P(runs in only one event) = 0.6857

d) P(runs in two events) = 0.1143

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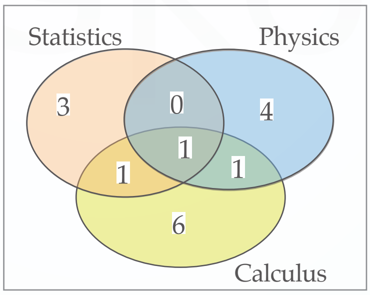
**56.** a)




b) P(calc and / or stat, not phy) = 0.625

c) P(more than one subject) = 0.1875

d) 81.25% teach only one subject

e) P(teach another subject given they teach calculus) = 0.3333

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**57.** a) 0.72

b) 0.56

c) 0.24

d) 0.27

e) 0.4231

f) 0.08

g) 0.4681

h) 0.61

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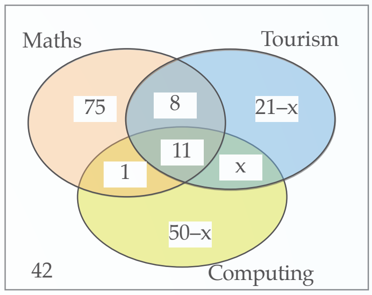
**58.**




a) $95 + (21 - x) + x + (50 - x) = 158$
$x = 8$

b) $P(T \text{ or } C) = \frac{83}{200}(0.415)$

c) $P(M \text{ and } T) = \frac{19}{200}(0.095)$

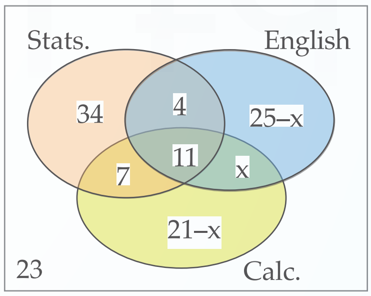
$P(M) \times P(T) = \frac{95}{200} \times \frac{40}{200}$

$= \frac{19}{200}(0.095)$

$= P(M \text{ and } T)$

Therefore independent.

**59.** a)




$56 + (21 - x) + x + (25 - x) = 102$
$x = 0$
$P(C \text{ and } E) = 0$

b) $P(\text{another} \mid S) = \frac{22}{56}(0.3929)$

c) That students in these three classes all come from Year 13. That Year 12 students are excluded from these figures.